ENRICHMENT OF CATEGORIES OF ALGEBRAS AND MODULES

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Abstract. We study the universal measuring coalgebras $P(A, B)$ of Sweedler and the universal measuring comodules $Q(M, N)$ of Batchelor. We show that these universal objects exist in a very general context. We provide a detailed proof of an observation of Wraith, that the $P(A, B)$ are the hom-objects of an enrichment of algebras in coalgebras. We show also that the $Q(M, N)$ provide an enrichment of the global category of modules in the global category of comodules.

1. Introduction

The notion of the universal measuring coalgebra $P(A, B)$ for algebras A and B was first introduced by Sweedler in [13]. The elements of $P(A, B)$ can be thought of as generalized maps from A to B. Examples of this point of view are given by Marjorie Batchelor in [3]. In the early 1970's, Gavin Wraith suggested that the coalgebra $P(A, B)$ can be used to give an enrichment of the category of algebras in the category of coalgebras. In this paper, we work through this fact in detail, since there is no treatment of Wraith's idea in the literature. We also extend his idea to give an enrichment of the global category of modules **Mod** in the global category of comodules **Comod**, using the measuring comodules, introduced by Batchelor in [3].

This paper is motivated by an issue concerning the two classic notions of enrichment and fibrations. There is no evident notion of fibration in the enriched setting. Here we consider the well-known fibration of the global category of modules **Mod** over the category of R-algebras $\mathbf{Alg}_R$ for R a commutative ring, and alongside it the opfibration of the category of comodules **Comod** over the category of R-coalgebras $\mathbf{Coalg}_R$. Since $\mathbf{Alg}_R$ and **Mod** are enriched in $\mathbf{Coalg}_R$ and **Comod** respectively, we would like to see this situation as some kind of *enriched fibration*.

Section 2 gives the background for the development of the following sections. The basic facts about categories of monoids/comonoids and modules/comodules in monoidal categories are presented, with particular emphasis on the symmetric monoidal category $\mathbf{Mod}_R$ for a commutative ring R. Additionally, we discuss locally presentable categories which are a useful context in which to frame later constructions. General references on these subjects for our purposes are [13], [12], [4] and [2]. Finally, we recall parts of the theory of the action of a monoidal category $\mathcal{V}$ on an ordinary category $\mathcal{D}$, which lead to an enrichment of $\mathcal{D}$ in $\mathcal{V}$, as in [6]. This is actually a special case of the more general result discussed in [5], that there is an equivalence between the 2-category of tensored $\mathcal{W}$-categories and the 2-category of $\mathcal{W}$-representations, for $\mathcal{W}$ a right-closed bicategory.

In Section 3, we consider the existence of the universal measuring coalgebra. The question that motivated the definition of measuring coalgebras is under which conditions, for A, B k-algebras and C a k-coalgebra (k a field), the linear map $\rho \in \mathrm{Hom}(A, \mathrm{Hom}(C, B))$ corresponding under the usual tensor-hom adjunction to $\sigma \in \mathrm{Hom}(C \otimes A, B)$ in $\mathbf{Vect_k}$, is actually an algebra map. In [13], Sweedler defines

what it means for a linear map $C \otimes A \xrightarrow{\sigma} B$ to *measure*, and so gives a category of measuring coalgebras (σ, C); he also gives a concrete construction of a terminal object $P(A,B)$ in this category, defined by the natural bijections

$$\mathbf{Alg}(A, \mathrm{Hom}(C,B)) \cong \{\sigma \in \mathrm{Hom}(C \otimes A, B) | \sigma \text{ measures}\} \cong \mathbf{Coalg}(C, P(A,B)),$$

called the *universal measuring coalgebra*. We identify the more general categorical ideas underlying this development, and prove the existence of the object $P(A,B)$ in a broader context. In particular, our construction covers the case of $\mathbf{Mod}_R$ for a commutative ring R. The class of *admissible* monoidal categories and results from [10] and [11] play an important role in this process, and in particular the natural isomorphism defining the object $P(A,B)$ in $\mathbf{Coalg}_R$ is also provided by [10, Proposition 4].

In Section 4, we combine the results of the two previous sections and we prove in our general context the result of Wraith, that there is an enrichment of the ordinary category $\mathbf{Alg}_R$ in the category $\mathbf{Coalg}_R$, with hom-objects $P(A,B)$.

In Section 5, we define the *global category of modules* and *comodules*, **Mod** and **Comod**. These two categories arise via the Grothendieck construction, corresponding to the functors sending an algebra (respectively coalgebra) to the category of its modules (respectively comodules) in $\mathbf{Mod}_R$, and they have nice categorical properties. In particular, as observed by Wischnewsky at the end of [14], the category **Comod** is comonadic over the simple product category $\mathbf{Mod}_R \times \mathbf{Coalg}_R$.

In Section 6, the universal measuring comodule $Q(M,N)$ is defined, via a natural isomorphism similar to the one defining the universal measuring algebra, $\mathbf{Comod}(X, Q(M,N)) \cong \mathbf{Mod}(M, \mathrm{Hom}(X,N))$. Applications of measuring comodules (as well as measuring coalgebras) can be found in [3]. We give a proof of existence of $Q(M,N)$ again in our general setting. This proof is not a direct generalization of the proof of existence of $P(A,B)$. Rather it makes detailed use of the internal structure of **Comod** and **Mod**.

In Section 7, in an analogous way to Section 4, we show how the global category **Mod** is enriched in the symmetric monoidal closed category **Comod**, with hom-objects $Q(M,N)$. Hence, at this point we have established the situation where the domain and codomain of the fibration $\mathbf{Mod} \to \mathbf{Alg}_R$ are enriched over the domain and codomain of the opfibration $\mathbf{Comod} \to \mathbf{Coalg}_R$.

In Section 8, we additionally prove that **Comod** is monoidal closed, and we sketch how, when we work in $\mathbf{Vect}_k$ for a field k, the existence of the measuring comodule is more straightforward than in the general case.

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2. Background

2.1. Categories of Monoids and Comonoids. The categories of monoids and comonoids for a monoidal category $\mathcal{V}$ are defined in the usual way. For example, $\mathbf{Comon}(\mathcal{V})$ has as objects triples (C, Δ, ϵ) where $C \in \mathrm{ob}\mathcal{V}$, $\Delta : C \to C \otimes C$ is the *comultiplication* and $\epsilon : C \to I$ is the *counit*, such that the diagrams

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "1\otimes\Delta"] \\ C \otimes C \arrow[r, "\Delta\otimes 1"] & C \otimes C \otimes C \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} I \otimes C & C \otimes C \arrow[l, "\epsilon\otimes 1"'] \arrow[r, "1\otimes\epsilon"] & C \otimes I \arrow[dl, "r_C"] \\ & C \arrow[u, "\Delta"] \arrow[ul, leftarrow, "l_C"] & \end{tikzcd}$$

commute, and has as morphisms $(C, \Delta, \epsilon) \to (C', \Delta', \epsilon')$ arrows $f : C \to C'$ in $\mathcal{V}$ such that

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "f"'] & C \otimes C \arrow[d, "f \otimes f"] \\ C' \arrow[r, "\Delta'"] & C' \otimes C' \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} C \arrow[r, "\epsilon"] \arrow[d, "f"'] & I \\ C' \arrow[ur, "\epsilon'"'] & \end{tikzcd}$$

commute. The category of monoids $\mathbf{Mon}(\mathcal{V})$ in a monoidal category is defined dually, with objects (A, m, η) where $A \in \mathrm{ob}\mathcal{V}$, $m : A \otimes A \to A$ is the *multiplication* and $\eta : I \to A$ is the *unit*.

Both the categories of monoids and comonoids of a symmetric monoidal category $\mathcal{V}$ are themselves symmetric monoidal categories, the tensor product and the symmetry inherited from $\mathcal{V}$. For example, $\mathbf{Mon}(\mathcal{V})$ is monoidal via the monoid structure

$$A \otimes B \otimes A \otimes B \xrightarrow{1 \otimes s \otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B$$

on $A \otimes B$ for $A, B \in \mathbf{Mon}(\mathcal{V})$, where s is the symmetry in $\mathcal{V}$.

There are obvious forgetful functors $\mathbf{Mon}(\mathcal{V}) \to \mathcal{V}$, $\mathbf{Comon}(\mathcal{V}) \to \mathcal{V}$, and these often have a left/right adjoint respectively, sending an object of $\mathcal{V}$ to the free monoid/cofree comonoid on that object. There exist various conditions on $\mathcal{V}$ that guarantee the existence of free monoids. For example, if $\mathcal{V}$ has countable coproducts which are preserved by the tensor product (on either side), then the free monoid on $X \in \mathrm{ob}\mathcal{V}$ is given by $\coprod_{n \in \mathbb{N}} X^{\otimes n}$.

Recall that given two (ordinary) adjoint functors $F \dashv G$ between monoidal categories, colax monoidal structures on the left adjoint F correspond bijectively, via mates under the adjunction, to lax monoidal structures on the right adjoint G. This is a well-known result, coming from the so-called *doctrinal adjunction* [7].

For example, in a symmetric monoidal closed category $\mathcal{V}$, since $- \otimes A \dashv [A, -]$ for all A, the bifunctor $[-, -] : \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}$ is the *parametrized adjoint* (see [9]) of the tensor product functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$. Since the tensor product is a (strong) monoidal functor, hence lax and colax, via

$$\phi_{(A,B),(A',B')} : A \otimes B \otimes A' \otimes B' \xrightarrow{\sim} A \otimes A' \otimes B \otimes B' \quad \text{and} \quad \phi_0 : I \xrightarrow{\sim} I \otimes I,$$

we get:

Corollary 2.1. *In a symmetric monoidal closed category $\mathcal{V}$, the internal hom functor $[-, -] : \mathcal{V}^{\mathrm{op}} \otimes \mathcal{V} \to \mathcal{V}$ is a lax monoidal functor, with lax structure maps $\psi_{(A,B),(A',B')} : [A, B] \otimes [A', B'] \to [A \otimes A', B \otimes B']$, $\psi_0 : I \to [I, I]$ corresponding under the adjunction $- \otimes X \dashv [X, -]$ to $[A, B] \otimes [A', B'] \otimes A \otimes A' \xrightarrow{1 \otimes s \otimes 1} [A, B] \otimes A \otimes [A', B'] \otimes A' \xrightarrow{e \otimes e} B \otimes B'$ and $I \otimes I \xrightarrow{\sim} I$.*

An important property of lax monoidal functors is that they map monoids to monoids. In particular, the internal hom for a symmetric monoidal closed category $\mathcal{V}$ takes monoids to monoids, and since we have $\mathbf{Mon}(\mathcal{V}^{\mathrm{op}} \times \mathcal{V}) \cong \mathbf{Comon}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Mon}(\mathcal{V})$, we obtain a functor

$$\begin{aligned} \mathbf{Mon}([-, -]) = H : \mathbf{Comon}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Mon}(\mathcal{V}) &\longrightarrow \mathbf{Mon}(\mathcal{V}) \\ (C, A) &\longmapsto [C, A]. \end{aligned} \tag{2.1}$$

The concrete content of this observation is that whenever C is a comonoid and A a monoid, the object $[C, A]$ is endowed with the structure of a monoid, with unit $I \to [C, A]$ which is the transpose, under $- \otimes C \dashv [C, -]$, of $C \xrightarrow{\epsilon} I \xrightarrow{\eta} A$, and with

multiplication $[C,A]\otimes[C,A]\to[C,A]$ the transpose of

$$\begin{tikzcd}
{[C,A]\otimes[C,A]\otimes C} \arrow[r, "1\otimes1\otimes\Delta"] \arrow[rrdd, dashed] & {[C,A]\otimes[C,A]\otimes C\otimes C} \arrow[r, "1\otimes s\otimes 1"] & {[C,A]\otimes C\otimes[C,A]\otimes C} \arrow[d, "e\otimes e"] \\
& & A\otimes A \arrow[d, "m"] \\
& & A.
\end{tikzcd}$$

2.2. **Categories of Modules and Comodules.** The categories of modules of a monoid and comodules of a comonoid in a monoidal category $\mathcal{V}$ are defined in the usual way. More precisely, for each monoid $(A,m,\eta)\in\mathbf{Mon}(\mathcal{V})$ there is a category $\mathbf{Mod}_{\mathcal{V}}(A)$ of (left) A-modules, with objects (M,μ), where M is an object in $\mathcal{V}$ and $\mu:A\otimes M\to M$ is the *action*, an arrow such that the diagrams

$$\begin{tikzcd}
A\otimes A\otimes M \arrow[r, "m\otimes1"] \arrow[d, "1\otimes\mu"'] & A\otimes M \arrow[d, "\mu"] \\
A\otimes M \arrow[r, "\mu"] & M
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
& A\otimes M \arrow[dr, "\mu"] & \\
I\otimes M \arrow[ur, "\eta\otimes1"] \arrow[rr, "l_M"] & & M
\end{tikzcd}
\tag{2.2}$$

commute, and morphisms $(M,\mu_M)\to(N,\mu_N)$ are arrows $u:M\to N$ in $\mathcal{V}$ such that

$$\begin{tikzcd}
A\otimes M \arrow[r, "1\otimes u"] \arrow[d, "\mu_M"'] & A\otimes N \arrow[d, "\mu_N"] \\
M \arrow[r, "u"] & N
\end{tikzcd}
\tag{2.3}$$

commutes. The category of (right) comodules $\mathbf{Comod}_{\mathcal{V}}(C)$ for a comonoid (C,Δ,ε) in $\mathbf{Comon}(\mathcal{V})$ is defined dually, with objects (X,δ), where X is an object in $\mathcal{V}$ and $\delta:X\to X\otimes C$ is the *coaction*, sastifying dual axioms. It is well-known that the forgetful functors $\mathbf{Mod}_{\mathcal{V}}(A)\xrightarrow{V_A}\mathcal{V}$ and $\mathbf{Comod}_{\mathcal{V}}(C)\xrightarrow{U_C}\mathcal{V}$ which simply forget the module action/comodule coaction, are respectively monadic/comonadic.

Each monoid arrow $f:A\to B$ determines a functor

$$\mathbf{Mod}(f)=f^\sharp:\mathbf{Mod}_{\mathcal{V}}(B)\to\mathbf{Mod}_{\mathcal{V}}(A) \tag{2.4}$$

sometimes called *restriction of scalars*, which makes every B-module N into an A-module $f^\sharp N$ via the action $A\otimes N\xrightarrow{f\otimes1}B\otimes N\xrightarrow{\mu}N$. Also, each B-module arrow becomes an A-module arrow, and so we have a commutative triangle of categories and functors

$$\begin{tikzcd}
\mathbf{Mod}_{\mathcal{V}}(B) \arrow[rr, "f^\sharp"] \arrow[dr, "V_B"'] & & \mathbf{Mod}_{\mathcal{V}}(A) \arrow[dl, "V_A"] \\
& \mathcal{V}. &
\end{tikzcd}
\tag{2.5}$$

Dually, each comonoid arrow $g:C\to D$ induces a functor

$$\mathbf{Comod}(g)=g_*:\mathbf{Comod}_{\mathcal{V}}(C)\to\mathbf{Comod}_{\mathcal{V}}(D) \tag{2.6}$$

called *corestriction of scalars*, which makes every C-comodule X into a D-comodule g_*X via the coaction $X\xrightarrow{\delta}X\otimes C\xrightarrow{1\otimes g}X\otimes D$. The respective commutative triangle is

$$\begin{tikzcd}
\mathbf{Comod}_{\mathcal{V}}(C) \arrow[rr, "g_*"] \arrow[dr, "U_C"'] & & \mathbf{Comod}_{\mathcal{V}}(D) \arrow[dl, "U_D"] \\
& \mathcal{V}. &
\end{tikzcd}
\tag{2.7}$$

By the above commutative triangles, where the legs are monadic and comonadic respectively, $f^\sharp$ is always continuous and g_* always cocontinuous.

We saw in the previous section how any lax monoidal functor $F : \mathcal{V} \to \mathcal{W}$ induces a functor $\mathbf{Mon}(F) : \mathbf{Mon}(\mathcal{V}) \to \mathbf{Mon}(\mathcal{W})$. Furthemore, this carries over to the categories of modules. If $M \in \mathbf{Mod}_{\mathcal{V}}(A)$ for a monoid A, then $FM \in \mathbf{Mod}_{\mathcal{W}}(FA)$, via the action

$$FA \otimes FM \xrightarrow{\phi_{A,M}} F(A \otimes M) \xrightarrow{F\mu} FM.$$

Again, we can apply this to the lax monoidal internal hom in a symmetric monoidal closed category $\mathcal{V}$, hence there is an induced functor for $C \in \mathbf{Comon}(\mathcal{V})$ and $A \in \mathbf{Mon}(\mathcal{V})$,

$$\begin{aligned} \mathbf{Comod}_{\mathcal{V}}(C)^{\mathrm{op}} \times \mathbf{Mod}_{\mathcal{V}}(A) &\longrightarrow \mathbf{Mod}_{\mathcal{V}}([C,A]) \\ (X,M) &\longmapsto [X,M]. \end{aligned} \tag{2.8}$$

Concretely, this means that whenever (X,δ) is a C-comodule and (M,μ) is an A-module, the object $[X,M]$ obtains the structure of a $[C,A]$-module, with action $[C,A] \otimes [X,M] \to [X,M]$ which is the transpose, under $- \otimes X \dashv [X,-]$, of

$$\begin{tikzcd} {[C,A] \otimes [X,M] \otimes X} \arrow[r, "1\otimes 1\otimes \delta"] \arrow[ddrr, dashed] & {[C,A] \otimes [X,M] \otimes X \otimes C} \arrow[r, "1\otimes s\otimes 1"] & {[C,A] \otimes C \otimes [X,M] \otimes X} \arrow[d, "e_A\otimes e_M"] \\ & & A \otimes M \arrow[d, "\mu"] \\ & & M. \end{tikzcd} \tag{2.9}$$

2.3. **Admissible Categories.** We mentioned earlier that the existence of the free monoid/cofree comonoid functor depends on specific assumptions on the category $\mathcal{V}$. We now consider a class of monoidal categories which provides good results regarding the formation of limits and colimits of the categories of monoids and comonoids, as well as properties which permit the existence of various adjunctions.

Recall (see [2]) that a *locally λ-presentable* category $\mathcal{K}$ is a cocomplete category which has a set $\mathcal{A}$ of λ-presentable objects such that every object is a λ-filtered colimit of objects from $\mathcal{A}$. In such a category $\mathcal{K}$, all λ-presentable objects have a set of representatives of isomorphism classes of objects. Any such set is denoted by $\mathbf{Pres}_{\lambda}\mathcal{K}$ and is a small dense full subcategory of $\mathcal{K}$, hence also a strong generator. Other useful properties of locally presentable categories are completeness, wellpoweredness and co-wellpoweredness.

Following the approach of [11], consider the class of *admissible* monoidal categories, i.e. locally presentable symmetric monoidal categories $\mathcal{V}$, such that for each object C, the functor $C \otimes -$ preserves filtered colimits. Examples of this class of categories is the category $\mathbf{Mod}_R$ for a commutative ring R, every locally presentable category with respect to cartesian products and every symmetric monoidal closed category that is locally presentable.

Then, for an admissible monoidal category $\mathcal{V}$, it is shown in [11] that $\mathbf{Mon}(\mathcal{V})$ is monadic over $\mathcal{V}$ and locally presentable itself and also $\mathbf{Comon}(\mathcal{V})$ is a locally presentable category and comonadic over $\mathcal{V}$.

The following simple adjoint functor theorem, identified by Max Kelly, will be used repeatedly.

Theorem 2.1. [8, 5.33] *If the cocomplete $\mathcal{C}$ has a small dense subcategory, every cocontinuous $S : \mathcal{C} \to \mathcal{B}$ has a right adjoint.*

As an application of this theorem, consider the category $\mathbf{Comon}(\mathcal{V})$ for a locally presentable symmetric monoidal closed $\mathcal{V}$. Then:

- $\mathbf{Comon}(\mathcal{V})$ is cocomplete.

- $\mathbf{Comon}(\mathcal{V})$ has a small dense subcategory, $\mathbf{Pres}_\lambda\mathbf{Comon}(\mathcal{V})$, since it is locally presentable.
- The functor $-\otimes C : \mathbf{Comon}(\mathcal{V}) \to \mathbf{Comon}(\mathcal{V})$ is cocontinuous, via the commutative diagram
$$\begin{tikzcd} \mathbf{Comon}(\mathcal{V}) \arrow[r, "-\otimes C"] \arrow[d, "F"'] & \mathbf{Comon}(\mathcal{V}) \arrow[d, "F"] \\ \mathcal{V} \arrow[r, "-\otimes FC"'] & \mathcal{V} \end{tikzcd}$$
, where $(-\otimes FC)\circ F$ is cocontinuous and the forgetful F creates colimits.

Therefore the functor $-\otimes C$ has a right adjoint, and hence the following result holds (see also [11, 3.2]):

Proposition 2.1. *If $\mathcal{V}$ is a locally presentable symmetric monoidal closed category, the category of comonoids $\mathbf{Comon}(\mathcal{V})$ is a monoidal closed category as well.*

2.4. **The category $\mathbf{Mod}_R$.** Consider the category $\mathbf{Mod}_R$ of R-modules and R-module maps, for R a commutative ring. It is of course a symmetric monoidal category, with the usual tensor product of R-modules. It is also monoidal closed, by the well-known adjunction $\begin{tikzcd} \mathbf{Mod}_R \arrow[r, shift left, "-\otimes N"] \arrow[r, phantom, "\perp"] & \mathbf{Mod}_R \arrow[l, shift left, "\mathrm{Hom}_R(N,-)"] \end{tikzcd}$, and moreover a locally presentable category, complete and cocomplete.

The categories of monoids and comonoids in $\mathbf{Mod}_R$ are $\mathbf{Mon}(\mathbf{Mod}_R) = \mathbf{Alg}_R$ and $\mathbf{Comon}(\mathbf{Mod}_R) = \mathbf{Coalg}_R$. Based on (2.1), the internal hom of the category induces the functor

$$\begin{aligned} \mathrm{Hom}_R : \mathbf{Coalg}_R^{\mathrm{op}} \times \mathbf{Alg}_R &\longrightarrow \mathbf{Alg}_R \\ (C, A) &\longmapsto \mathrm{Hom}_R(C, A). \end{aligned} \tag{2.10}$$

This is the same as the well-known fact that for C an R-coalgebra and A an R-algebra, $\mathrm{Hom}_R(C, A)$ obtains the structure of an R-algebra under the *convolution product*

$$(f * g)(c) = \sum f(c_1)g(c_2) \quad \text{and} \quad 1 = \eta\circ\epsilon. \tag{2.11}$$

Now, since $\mathbf{Mod}_R$ is an admissible category in the sense of the previous section, we deduce that $\mathbf{Alg}_R$ is a locally presentable category and monadic over $\mathbf{Mod}_R$, and $\mathbf{Coalg}_R$ is comonadic over $\mathbf{Mod}_R$, locally presentable and monoidal closed.

Denote by $\mathbf{Comod}_C$ the category of R-modules which have a C-comodule structure for an R-coalgebra C, and respectively $\mathbf{Mod}_A$ the category of A-modules. We know that these categories possess many useful properties (see e.g. [14]). In particular, apart from the facts that $\mathbf{Mod}_A$ is monadic and $\mathbf{Comod}_C$ is comonadic over $\mathbf{Mod}_R$, we also have that $\mathbf{Comod}_C$ is complete, wellpowered and co-wellpowered, has a generator and a cogenerator and is locally presentable.

Regarding the restriction and corestriction of scalars in this case, for $\mathcal{V} = \mathbf{Mod}_R$, $f : A \to B$ in $\mathbf{Alg}_R$ and $g : C \to D$ in $\mathbf{Coalg}_R$, the triangles (2.5), (2.7) become

$$\begin{tikzcd}[column sep=small] \mathbf{Mod}_B \arrow[rr, "f^\sharp"] \arrow[dr, "V_B"'] & & \mathbf{Mod}_A \arrow[dl, "V_A"] \\ & \mathbf{Mod}_R & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small] \mathbf{Comod}_C \arrow[rr, "g_*"] \arrow[dr, "U_C"'] & & \mathbf{Comod}_D \arrow[dl, "U_D"] \\ & \mathbf{Mod}_R & \end{tikzcd}.$$

Since $\mathbf{Mod}_R$ is a symmetric monoidal closed category, with all limits and colimits, we obtain a pair of adjoints $\begin{tikzcd} \mathbf{Mod}_B \arrow[r, "f^\sharp" description] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, phantom, shift left=3, "\perp"] & \mathbf{Mod}_A \arrow[l, bend right=40, "f_\sharp"'] \arrow[l, bend left=40, "\tilde{f}"] \end{tikzcd}$ for the restriction of

scalars, the left adjoint $f_\sharp \dashv f^\sharp$ given by $f_\sharp \cong B \otimes_A - : \mathbf{Mod}_A \to \mathbf{Mod}_B$ where B is regarded as a left-B right-A bimodule, and the right adjoint $f^\sharp \dashv \tilde{f}$ given by $\tilde{f} \cong \mathrm{Hom}_A(B,-) : \mathbf{Mod}_A \to \mathbf{Mod}_B$ where B is regarded as a left-A right-B bimodule.

As for the corestriction of scalars, because of the properties of $\mathbf{Comod}_C$ mentioned above, there exists a right adjoint $g_* \dashv g^*$, which is given, when C is a flat R-coalgebra, by $g^* \cong -\Box_D C : \mathbf{Comod}_D \to \mathbf{Comod}_C$, where C is regarded as a left-D right-C bicomodule and $\Box$ is the cotensor product (e.g. see [14]).

Next, we can apply (2.8) to $\mathbf{Mod}_R$. For C an R-coalgebra and A an R-algebra, the induced map, denoted by Hom, is

$$\begin{aligned} \mathrm{Hom} : \mathbf{Comod}_C^{\mathrm{op}} \times \mathbf{Mod}_A &\longrightarrow \mathbf{Mod}_{\mathrm{Hom}_R(C,A)} \\ (X,M) &\longmapsto \mathrm{Hom}_R(X,M) \end{aligned} \tag{2.12}$$

This means that whenever (X,δ) is a C-comodule and (M,μ) is an A-module, then $(\mathrm{Hom}_R(X,M),\mu')$ is a $\mathrm{Hom}_R(C,A)$-module, denoted just by $\mathrm{Hom}(X,M)$, the action μ' described as in (2.9).

Moreover, the functor $\mathrm{Hom}(-,M) : \mathbf{Comod}_C^{\mathrm{op}} \to \mathbf{Mod}_{\mathrm{Hom}_R(C,A)}$ for any $M \in \mathbf{Mod}_A$ is continuous, which is obvious by the following commutative square, where $\mathrm{Hom}_R(-,UM) \circ U^{\mathrm{op}}$ preserves limits and the forgetful V creates them:

$$\begin{tikzcd} \mathbf{Comod}_C^{\mathrm{op}} \arrow[r, "\mathrm{Hom}(-,M)"] \arrow[d, "U^{\mathrm{op}}"'] & \mathbf{Mod}_{\mathrm{Hom}_R(C,A)} \arrow[d, "V"] \\ \mathbf{Mod}_R^{\mathrm{op}} \arrow[r, "\mathrm{Hom}_R(-,UM)"] & \mathbf{Mod}_R. \end{tikzcd} \tag{2.13}$$

Similarly, for each D-comodule Y, the evident functor $-\otimes Y : \mathbf{Comod}_C \longrightarrow \mathbf{Comod}_{C\otimes D}$ is cocontinuous, as it is clear from the following commutative diagram:

$$\begin{tikzcd} \mathbf{Comod}_C \arrow[r, "-\otimes Y"] \arrow[d, "U"'] & \mathbf{Comod}_{C\otimes D} \arrow[d, "U"] \\ \mathbf{Mod}_R \arrow[r, "-\otimes UY"] & \mathbf{Mod}_R. \end{tikzcd} \tag{2.14}$$

2.5. **Actions of a Monoidal Category.** Recall that an *action* of a monoidal category $\mathcal{V} = (\mathcal{V},\otimes,I,a,l,r)$ on a category $\mathcal{D}$ is given by a functor $* : \mathcal{V}\times\mathcal{D} \to \mathcal{D}$ written $(X,D) \mapsto X*D$, a natural isomorphism with components $\alpha_{XYD} : (X\otimes Y)*D \xrightarrow{\sim} X*(Y*D)$, and a natural isomorphism with components $\lambda_D : I*D \xrightarrow{\sim} D$, satisfying the commutativity of the diagrams

$$\begin{tikzcd} ((X\otimes Y)\otimes Z)*D \arrow[r, "\alpha"] \arrow[d, "a*1"'] & (X\otimes Y)*(Z*D) \arrow[r, "\alpha"] & X*(Y*(Z*D)) \\ (X\otimes(Y\otimes Z))*D \arrow[rr, "\alpha"] & & X*((Y\otimes Z)*D), \arrow[u, "1*\alpha"'] \end{tikzcd} \tag{2.15}$$

$$\begin{tikzcd} (I\otimes X)*D \arrow[rr, "\alpha"] \arrow[dr, "l*1"'] & & I*(X*D) \arrow[dl, "\lambda"] \\ & X*D, & \end{tikzcd} \tag{2.16}$$

$$\begin{tikzcd} (X\otimes I)*D \arrow[rr, "\alpha"] \arrow[dr, "r*1"'] & & X*(I*D) \arrow[dl, "1*\lambda"] \\ & X*D. & \end{tikzcd} \tag{2.17}$$

Remark 2.1. A monoidal category $\mathcal{V}$ is actually a *pseudomonoid* inside the monoidal category $(\mathbf{Cat}, \times, 1)$, so this action described above is exactly the definition of a *pseudoaction* of a pesudomonoid on an object of $\mathbf{Cat}$.

The most important fact here, explained in detail in [6], is that to give a category $\mathcal{D}$ and an action of a monoidal closed $\mathcal{V}$ with a right adjoint for each $- * D$ is to give a tensored $\mathcal{V}$-category:

Proposition 2.2. *Suppose that $\mathcal{V}$ is a monoidal category which acts on a category $\mathcal{D}$ via the bifunctor $H : \mathcal{V} \times \mathcal{D} \to \mathcal{D}$, and $H(-, B)$ has a right adjoint $H(-, B) \dashv F(B, -)$ for every $B \in \mathcal{D}$, with the natural isomorphism*

$$\mathcal{D}(H(X, B), D) \cong \mathcal{V}(X, F(B, D)). \tag{2.18}$$

Then, we can enrich $\mathcal{D}$ in $\mathcal{V}$, in the sense that there is a $\mathcal{V}$-category $\mathcal{K}$, with the same objects as $\mathcal{D}$, hom-objects $\mathcal{K}(A, B) = F(A, B)$ and underlying category $\mathcal{K}_o = \mathcal{D}$.

The proof that there exists a composition functor $M : F(B, C) \otimes F(A, B) \to F(A, C)$ and identity elements $j_A : I \to F(A, A)$ satisfying the usual axioms of enriched categories relies on just the correspondence of arrows under the adjunction (2.18) and the action properties. We denote the $\mathcal{V}$-category $\mathcal{K}$ again by $\mathcal{D}$.

When $\mathcal{V}$ is monoidal closed, we get a natural isomorphism $\mathcal{D}(H(X, B), D) \cong [X, F(B, D)]$, and so this $\mathcal{V}$-enriched representation defines the *tensor product* (see [8]) in $\mathcal{D}$ of X and B, namely $H(X, B)$.

As a special case of this result, consider the situation when $\mathcal{V} = \mathcal{C}$, $\mathcal{D} = \mathcal{A}^{\mathrm{op}}$, and the action is the bifunctor $H : \mathcal{C} \times \mathcal{A}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$. Note that, via $H(-, B) \dashv F(B, -)$, the functor F is actually the parametrized adjoint of H, so it is also a bifunctor $F : \mathcal{A} \times \mathcal{A}^{\mathrm{op}} \to \mathcal{C}$. Denote this F as $P : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{C}$ and so the natural isomorphism (2.18) of the adjunction $H(-, B) \dashv P(-, B)$ becomes

$$\mathcal{A}^{\mathrm{op}}(H(C, B), A) \cong \mathcal{C}(C, P(A, B)).$$

Corollary 2.2. *When we have an action $H : \mathcal{C} \times \mathcal{A}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$ of the monoidal $\mathcal{C}$ on the category $\mathcal{A}^{\mathrm{op}}$ along with an adjunction* $\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "H(-,B)"] \arrow[r, phantom, "\perp"] & \mathcal{A}^{\mathrm{op}} \arrow[l, shift left, "P(-,B)"] \end{tikzcd}$ *for each B, then $\mathcal{A}^{\mathrm{op}}$ is enriched in $\mathcal{C}$ with hom-objects $\mathcal{A}^{\mathrm{op}}(A, B) = P(B, A)$, and $H(C, B)$ is the tensor product of C and B.*

Moreover, when the monoidal category $\mathcal{C}$ is symmetric, then $\mathcal{A} = (\mathcal{A}^{\mathrm{op}})^{\mathrm{op}}$ is also enriched in $\mathcal{C}$, with the same objects and hom-objects $\mathcal{A}(A, B) = \mathcal{A}^{\mathrm{op}}(B, A)$. Hence,

Corollary 2.3. *If $\mathcal{C}$ is symmetric and the above conditions hold, then $\mathcal{A}$ is enriched in $\mathcal{C}$, with hom-objects $\mathcal{A}(A, B) = P(A, B)$.*

3. The existence of the universal measuring coalgebra

As we mentioned in the introduction, we explore the existence of an object $P(A, B)$ and a natural isomorphism

$$\mathbf{Alg}_k(A, \mathrm{Hom}_k(C, B)) \cong \mathbf{Coalg}_k(C, P(A, B)) \tag{3.1}$$

defining the universal measuring coalgebra, but for general categories of monoids and comonoids in some monoidal category $\mathcal{V}$. Since the functor $\mathrm{Hom}_k(-, -)$ is in fact the internal hom, algebras are the monoids and coalgebras are the comonoids in $\mathbf{Vect_k}$, this evidently comes down to the existence of an adoint of the internal hom functor applied on comonoids in the first variable.

Consider a symmetric monoidal closed category $\mathcal{V}$. We saw in the previous section that a functor $H : \mathbf{Comon}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Mon}(\mathcal{V}) \to \mathbf{Mon}(\mathcal{V})$ is induced by the internal hom (see (2.1)). In order to find a right adjoint for

$$H(-,B)^{\mathrm{op}} : \mathbf{Comon}(\mathcal{V}) \to \mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$$

we can use Theorem 2.1. For that we need $H(-,B)^{\mathrm{op}}$ to be a cocontinuous functor and $\mathbf{Comon}(\mathcal{V})$ to be a cocomplete category with a small dense subcategory.

For example, in the case of $\mathcal{V} = \mathbf{Vect_k}$, one can easily see that the above conditions hold: $\mathbf{Coalg_k}$ is a cocomplete category and has a small dense subcategory, namely the coalgebras which are finitely dimensional as vector spaces over k, and the hom functor is continuous. So we recover (3.1).

We saw in Section 2.3 that if we consider the class of admissible monoidal categories, we already know that $\mathbf{Comon}(\mathcal{V})$ is comonadic over $\mathcal{V}$, hence cocomplete, and is also a locally presentable category, hence it has a small dense subcategory. Moreover, the diagram

$$\begin{tikzcd} \mathbf{Comon}(\mathcal{V})^{\mathrm{op}} \arrow[r, "H(-,B)"] \arrow[d, "F_1^{\mathrm{op}}"'] & \mathbf{Mon}(\mathcal{V}) \arrow[d, "F_2"] \\ \mathcal{V}^{\mathrm{op}} \arrow[r, "{[-,F_2B]}"] & \mathcal{V} \end{tikzcd} \tag{3.2}$$

commutes, where F_1, F_2 are the respective forgetful functors. The functor $[-,F_2B]$ is continuous as the right adjoint of $[-,F_2B]^{\mathrm{op}}$, so the composite $[-,F_2B] \circ F_1^{\mathrm{op}}$ preserve all limits, and also the monadic F_2 creates limits. Therefore $H(-,B)$ is continuous, and so its opposite functor $H(-,B)^{\mathrm{op}}$ is cocontinuous.

Since all the required conditions of the theorem are satisfied, we obtain the following result.

Proposition 3.1. *For an admissible monoidal category $\mathcal{V}$, the functor $H(-,B)^{\mathrm{op}}$ has a right adjoint, namely $P(-,B) : \mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \to \mathbf{Comon}(\mathcal{V})$, and a natural isomorphism*

$$\mathbf{Comon}(\mathcal{V})(C, P(A,B)) \cong \mathbf{Mon}(\mathcal{V})^{\mathrm{op}}(H(C,B)^{\mathrm{op}}, A) \tag{3.3}$$

is established.

In particular, since the category $\mathbf{Mod}_R$ is an admissible monoidal category, we can apply the above to get an adjunction $\begin{tikzcd} \mathbf{Coalg}_R \arrow[r, shift left, "\mathrm{Hom}_R(-,B)^{\mathrm{op}}"] \arrow[r, phantom, "\perp"] & \mathbf{Alg}_R^{\mathrm{op}} \arrow[l, shift left, "P(-,B)"] \end{tikzcd}$ given by

$$\mathbf{Coalg}_R(C, P(A,B)) \cong \mathbf{Alg}_R(A, \mathrm{Hom}_R(C,B)). \tag{3.4}$$

We call the object $P(A,B)$ in $\mathbf{Coalg}_R$ the *universal measuring coalgebra*. We obtain a bifunctor

$$P : \mathbf{Alg}_R^{\mathrm{op}} \times \mathbf{Alg}_R \to \mathbf{Coalg}_R \tag{3.5}$$

such that the above isomorphism is natural in all three variables.

Remark 3.1. The functor $\mathrm{Hom}_k(-,k)$ for a field k is the so-called 'dual algebra functor', taking any coalgebra C to its dual C^*=$\mathrm{Hom}_k(C,k)$, which has a natural structure of an algebra. It is well-known that, since A^* for an algebra A in general fails to have a coalgebra structure, we can define

$$A^0 = \{g \in A^* | ker g \text{ contains a cofinite ideal}\}$$

which turns out to be a coalgebra, so that the functors $(\ )^0$ and $(\ )^*$ are adjoint to one another. The adjunction (3.4) then, for $B = k$, produces the well-known isomorphism

$$\mathbf{Coalg}_k(C, A^0) \cong \mathbf{Alg}_k(A, C^*)$$

therefore $P(A,k) \cong A^0$. So actually (3.4) generalizes the dual algebra functor adjunction for R a commutative ring.

We now proceed to the statement and proof of a lemma that connects the above adjunction (3.4) with the usual $-\otimes_R C \dashv \mathrm{Hom}_R(C,-)$ for arbitrary R-modules.

Lemma 3.1. *Suppose we have an R-algebra map $f : A \to \mathrm{Hom}_R(C,B)$, $A, B \in \mathbf{Alg}_R$, $C \in \mathbf{Coalg}_R$. If it corresponds to $\bar{f} : A \otimes C \to B$ under $-\otimes C \dashv \mathrm{Hom}_R(C,-)$ and to $\hat{f} : C \to P(A,B)$ under $\mathrm{Hom}_R(-,B)^{\mathrm{op}} \dashv P(-,B)$, then $\bar{f} = (\alpha \otimes \hat{f}) \circ e$, where e is the evaluation and α the unit of the second adjunction.*

Proof.

$$\begin{tikzcd}
 & \mathrm{Hom}_R(P(A,B),B)\otimes C \arrow[rr, "1\otimes\hat{f}"] \arrow[ddrr, "\mathrm{Hom}_R(\hat{f},1)\otimes 1"] & & \mathrm{Hom}_R(P(A,B),B)\otimes P(A,B) \arrow[dr, "e_B"] & \\
A\otimes C \arrow[ur, "\alpha\otimes 1"] \arrow[drrr, "f\otimes 1"'] \arrow[rrrr, dashed, "\bar{f}"] & & & & B \\
 & & & \mathrm{Hom}_R(C,B)\otimes C \arrow[ur, "e_B"'] &
\end{tikzcd}$$

By inspection of the diagram, we can see that the left part is the commutative diagram which gives f through its transpose map $\hat{f}$ under $\mathrm{Hom}_R(-,B)^{\mathrm{op}} \dashv P(-,B)$, and the right part commutes by dinaturality of the counit $e^E_D : \mathrm{Hom}_R(D,E)\otimes D \to E$ of the parametrized adjunction $-\otimes - \dashv \mathrm{Hom}_R(-,-)$. □

4. Enrichment of algebras in coalgebras

Now that we have established the existence of the measuring coalgebra $P(A,B)$ in $\mathbf{Mod}_R$ through the isomorphism $\mathbf{Coalg}_R(C, P(A,B)) \cong \mathbf{Alg}_R(A, \mathrm{Hom}_R(C,B))$, we can combine this result with the theory of actions of monoidal categories and show that there exists a way to enrich the category $\mathbf{Alg}_R$ in the symmetric monoidal closed category $\mathbf{Coalg}_R$.

Remark 4.1. For any symmetric monoidal category $\mathcal{V}$, the internal hom bifunctor $[-,-] : \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}$ is an action of $\mathcal{V}^{\mathrm{op}}$ on $\mathcal{V}$, via the natural isomorphisms

$$\alpha_{XYZ} : [X\otimes Y, D] \xrightarrow{\sim} [X,[Y,Z]] \quad \text{and} \quad \lambda_D : [I,D] \xrightarrow{\sim} D.$$

Hence the functor $\mathrm{Hom}_R : \mathbf{Mod}_R^{\mathrm{op}} \times \mathbf{Mod}_R \to \mathbf{Mod}_R$ is an action (and $\mathrm{Hom}_R^{\mathrm{op}}$ is an action too).

If we take $\mathcal{A} = \mathbf{Alg}_R$ and $\mathcal{C} = \mathbf{Coalg}_R$ in the statement of Corollary 2.2, we do have a bifunctor

$$H = \mathrm{Hom}_R^{\mathrm{op}} : \mathbf{Coalg}_R \times \mathbf{Alg}_R^{\mathrm{op}} \to \mathbf{Alg}_R^{\mathrm{op}},$$

and $\mathrm{Hom}_R(-,B)^{\mathrm{op}}$ has a right adjoint $P(-,B)$ (by (3.4)). What remains to be shown is that the bifunctor $H = \mathrm{Hom}_R(-,-)$ is an *action* of $\mathbf{Coalg}_R^{\mathrm{op}}$ on $\mathbf{Alg}_R$ (so then also $\mathrm{Hom}_R^{\mathrm{op}}$ will be an action of $\mathbf{Coalg_R}$ on $\mathbf{Alg_R}^{\mathrm{op}}$). We are looking for two natural isomorphisms in $\mathbf{Alg}_R$

$$\alpha_{CDA} : \mathrm{Hom}_R(C\otimes D, A) \xrightarrow{\sim} \mathrm{Hom}_R(C, \mathrm{Hom}_R(D,A)),\ \ \lambda_A : \mathrm{Hom}_R(R,A) \xrightarrow{\sim} A$$

such that the diagrams (2.15), (2.16), (2.17) commute. But we already know that these isomorphisms exist in $\mathbf{Mod}_R$ and make the functor Hom_R into an action, by the Remark 4.1 above. So it is enough to see that these R-module isomorphisms in fact lift to R-algebra isomorphisms (since $\mathbf{Alg}_R$ is monadic over $\mathbf{Mod}_R$), where $\mathrm{Hom}_R(C\otimes D, A)$, $\mathrm{Hom}_R(C, \mathrm{Hom}_R(D,A))$ and $\mathrm{Hom}_R(R,A)$ are algebras via convolution (see (2.11)).

Furthemore, the diagrams which define an action commute, since they do for all R-modules. Therefore the bifunctor $\mathrm{Hom}_R : \mathbf{Coalg}_R^{\mathrm{op}} \times \mathbf{Alg}_R \to \mathbf{Alg}_R$ is indeed an action, and so Corollaries 2.2 and 2.3 apply on this case:

Proposition 4.1. *The category $\mathbf{Alg}_R^{\mathrm{op}}$ is enriched in $\mathbf{Coalg}_R$, with hom-objects $\mathbf{Alg}_R^{\mathrm{op}}(A,B) = P(B,A)$, and $\mathrm{Hom}_R(C,B)$ is the tensor product of C and B.*

Proposition 4.2. *The category $\mathbf{Alg}_R$ is enriched in $\mathbf{Coalg}_R$, with hom-objects $\mathbf{Alg}_R(A,B) = P(A,B)$.*

5. The categories **Comod** and **Mod**

We consider the *global category of comodules* and the *global category of modules* over a commutative ring R. In reality, both definitions are derived from the well-known fact that, via the Grothendieck construction, there is a correspondance between pseudofunctors $\mathcal{H} : \mathcal{B}^{\mathrm{op}} \to \mathbf{Cat}$ and cloven fibrations $P = \int \mathcal{H} : \mathcal{X} \to \mathcal{B}$, and also between functors $\mathcal{H}$ and split fibrations $\int \mathcal{H}$ (and dually for opfibrations). Hence, the two functors

$$\mathcal{H} : \mathbf{Alg}_R^{\mathrm{op}} \to \mathbf{Cat} \quad \text{and} \quad \mathcal{G} : \mathbf{Coalg}_R \to \mathbf{Cat}$$

which send an algebra A to $\mathbf{Mod}_A$ and a coalgebra C to $\mathbf{Comod}_C$, based on the well-behaved restriction and corestriction of scalars (see (2.4) and (2.6)), give rise to the global categories and the functors $V : \mathbf{Mod} \to \mathbf{Alg}_R$, $U : \mathbf{Comod} \to \mathbf{Coalg}_R$. The explicit definition of **Comod** is the following:

- Objects: all comodules X over all R-coalgebras C, denoted by X_C.
- Morphisms: if X is a C-comodule and Y is a D-comodule, a map between them is a pair $(k,g) : X_C \to Y_D$ where $\begin{cases} g_*X \xrightarrow{k} Y & \text{in } \mathbf{Comod}_D \\ C \xrightarrow{g} D & \text{in } \mathbf{Coalg}_R \end{cases}$.
- Composition: When we have two morphisms $X_C \xrightarrow{(k,g)} Y_D \xrightarrow{(l,h)} Z_E$, so $\begin{cases} g_*X \xrightarrow{k} Y & \text{in } \mathbf{Comod}_D \\ C \xrightarrow{g} D & \text{in } \mathbf{Coalg}_R \end{cases}$ and $\begin{cases} h_*Y \xrightarrow{l} Z & \text{in } \mathbf{Comod}_E \\ D \xrightarrow{h} E & \text{in } \mathbf{Coalg}_R \end{cases}$, then their composite $X_C \xrightarrow{(lk,hg)} Z_E$ is $\begin{cases} (hg)_*X \xrightarrow{lk} Z & \text{in } \mathbf{Comod}_E \\ C \xrightarrow{hg} E & \text{in } \mathbf{Coalg}_R \end{cases}$ where lk is given by the commutative triangle
$$\begin{tikzcd} h_*g_*X \arrow[r, dashed] \arrow[dr, "h_*k"'] & Z \\ & h_*Y \arrow[u, "l"'] \end{tikzcd}.$$

- Associativity: It holds due to the associativity of coalgebra and comodule morphisms.
- Identity: The map $\begin{cases} (1_C)_*X = X \xrightarrow{1_X} X & \text{in } \mathbf{Comod}_C \\ C \xrightarrow{1_C} C & \text{in } \mathbf{Coalg}_R \end{cases}$ works as the identity morphism $X_C \xrightarrow{(1_X,1_C)} X_C$ in this category.

There is an evident 'forgetful' functor $U : \mathbf{Comod} \to \mathbf{Coalg}_R$ which maps any comodule X_C to its coalgebra C, and any morphism to the coalgebra map of the pair. This functor is not faithful, since for each coalgebra map $UX_C \xrightarrow{g} UY_D$ we can choose more than one map in $\mathbf{Comod}_D(g_*X, Y)$, but it is full: any coalgebra map $g : UX_C \to UY_D$ can be written as $U(\varepsilon, g)$, for $\begin{cases} g_*g^*Y \xrightarrow{\varepsilon} Y & \text{in } \mathbf{Comod}_D \\ C \xrightarrow{g} D & \text{in } \mathbf{Coalg}_R \end{cases}$, where ε is the counit of the adjunction $g_* \dashv g^*$ (see Section 2.4).

We now explore some of the properties of this category. First of all, **Comod** is a symmetric monoidal category: if $X_C, Y_D \in \mathbf{Comod}$, then $X_C \otimes Y_D \in \mathbf{Comod}$, being an R-module with a coaction

$$X \otimes Y \xrightarrow{\delta_X \otimes \delta_Y} X \otimes C \otimes Y \otimes D \xrightarrow{\sim} X \otimes Y \otimes C \otimes D$$

over the coalgebra $C \otimes D$ ($\mathbf{Coalg}_R$ is monoidal), and the symmetry is inherited from $\mathbf{Mod}_R$. Notice that we also have $U(X_C \otimes Y_D) = C \otimes D = UX_C \otimes UY_D$, which means that the functor U has the structure of a *strict* monoidal functor.

Furthemore, **Comod** is a cocomplete category. This can be shown to be true either via the explicit construction of colimits in **Comod**, or as a corollary to a stronger result:

Proposition 5.1. *The functor* $G : \mathbf{Comod} \to \mathbf{Mod}_R \times \mathbf{Coalg}_R$*, given by* $X_C \mapsto (X, C)$*, is comonadic.*

Proof. Define the functor $H : \mathbf{Mod}_R \times \mathbf{Coalg}_R \to \mathbf{Comod}$ by $(V, D) \mapsto (V \otimes D)_D$ on objects and

$$\begin{tikzcd} (V,D) \arrow[r, mapsto] \arrow[d, "{(k,f)}"] & (V \otimes D)_D \arrow[d, "{(k\otimes f,f)}"] \\ (W,E) \arrow[r, mapsto] & (W \otimes E)_E \end{tikzcd}$$

on arrows. Notice that $V \otimes D$ is a D-comodule via $V \otimes D \xrightarrow{1\otimes\Delta} V \otimes D \otimes D$, and also the linear map $k \otimes f$ is in fact an E-comodule arrow $f_*(V \otimes D) \to W \otimes E$, so the mappings are well-defined. Then, this functor H is the right adjoint of G, via the natural bijection

$$\begin{aligned}(\mathbf{Mod}_R \times \mathbf{Coalg}_R)((X,C),(V,D)) &\cong \mathbf{Mod}_R(X,V) \times \mathbf{Coalg}_R(C,D) \\ &\cong \mathbf{Comod}(X_C, (V \otimes D)_D) \qquad (5.1)\end{aligned}$$

where $(X, C) = G(X_C)$ and $(V \otimes D)_D = H(V, D)$, as shown below:

i) Given a pair of maps $\begin{cases} k : X \to V & \text{in } \mathbf{Mod}_R \\ f : C \to D & \text{in } \mathbf{Coalg}_R \end{cases}$, we obtain the arrow $(\bar{k}, f)$: $X_C \to (V \otimes D)_D$ in **Comod**, where $\bar{k}$ as a linear map is $X \xrightarrow{\delta} X \otimes C \xrightarrow{k\otimes f} V \otimes D$.

ii) Given $X_C \xrightarrow{(l,g)} (V \otimes D)_D$ in **Comod**, i.e. $\begin{cases} g_*X \xrightarrow{l} V \otimes D & \text{in } \mathbf{Comod}_D \\ \quad C \xrightarrow{g} D & \text{in } \mathbf{Coalg}_R \end{cases}$, we get the pair of arrows $(X \xrightarrow{l} V \otimes D \xrightarrow{1\otimes\epsilon} V, C \xrightarrow{g} D)$ in $\mathbf{Mod}_R(X,V) \times \mathbf{Coalg}_R(C,D)$.

Moreover, these two directions are inverses to each other, therefore the bijection (5.1) is established and we obtain the adjunction $\begin{tikzcd} \mathbf{Comod} \arrow[r, shift left, "G"] \arrow[r, phantom, "\perp"] & \mathbf{Mod}_R \times \mathbf{Coalg}_R \arrow[l, shift left, "H"] \end{tikzcd}$.

Hence, a comonad is induced on $\mathbf{Mod}_R \times \mathbf{Coalg}_R$, given by the endofunctor GH which maps (V, D) to $(V \otimes D, D)$, the counit $\varepsilon : GH \Rightarrow id$ with components $\varepsilon_{(V,D)} : (V \otimes D, D) \xrightarrow{(1\otimes\epsilon,1)} (V, D)$ and the natural transformation $G\eta_H : GH \Rightarrow GHGH$ with components $G\eta_{H(V,D)} : (V \otimes D, D) \xrightarrow{(1\otimes\Delta,1)} (V \otimes D \otimes D, D)$. A coalgebra for this comonad $((V, D), \gamma)$ with $(V, D) \xrightarrow{\gamma} (V \otimes D, D)$ turns out to be exactly a D-comodule for each different $(D, \Delta, \epsilon) \in \mathbf{Coalg}_R$, and also a coalgebra arrow $((V, D), \gamma) \xrightarrow{(k,f)} ((W, E), \gamma')$ turns out to be the same as an E-comodule morphism $f_*V \to W$. Hence, the category of GH-coalgebras $(\mathbf{Mod}_R \times \mathbf{Coalg}_R)^{GH}$ is **Comod**. □

Corollary 5.1. *The category* **Comod** *is cocomplete, the functor* $G : \mathbf{Comod} \to \mathbf{Mod}_R \times \mathbf{Coalg}_R$ *creating all colimits.*

For the explicit construction of colimits in **Comod**, consider a diagram $F : J \to$ **Comod**. Since $\mathbf{Coalg}_R$ is cocomplete, the diagram $J \xrightarrow{F} \mathbf{Comod} \xrightarrow{U} \mathbf{Coalg}_R$ has a colimiting cocone $(UF_j \xrightarrow{\tau_j} \operatorname{colim}(UF) \,/\, j \in J)$ with

$$\begin{tikzcd} UF_j \arrow[r, "\tau_j"] \arrow[d, "UF\alpha"'] & \operatorname{colim}(UF) \\ UF_{j'} \arrow[ur, "\tau_{j'}"'] & \end{tikzcd}$$

commuting for any $j \xrightarrow{\alpha} j'$. Define a new diagram $G : J \to \mathbf{Comod}_{\operatorname{colim}(UF)}$ by

$$\begin{tikzcd} j \arrow[r, mapsto] \arrow[d, "\alpha"'] & (\tau_j)_* F_j = (\tau_{j'})_*(UF\alpha)_* F_j \arrow[d, "(\tau_{j'})_* F\alpha"] \\ j' \arrow[r, mapsto] & (\tau_{j'})_* F_{j'} \end{tikzcd}$$

. Since $\mathbf{Comod}_{\operatorname{colim}(UF)}$ is cocomplete, the above diagram has a colimiting cocone $((\tau_j)_*F_j \xrightarrow{\sigma_j} \operatorname{colim} G \,/\, j \in J)$, and also $U \operatorname{colim} G = \operatorname{colim}(UF)$.

To see that $\operatorname{colim} G$ is actually the colimit of the initial diagram F, notice that $(F_j \xrightarrow{(\sigma_j, \tau_j)} \operatorname{colim} G \,/\, j \in J)$ is already a cocone in **Comod** (the comodules being over the obvious coalgebras), and $\operatorname{colim} G$ has the respective universal property in **Comod**.

In a very similar way, we can define the category **Mod** of all modules over all R-algebras for a commutative ring R. A morphism in **Mod** between an A-module M and a B-module N is defined as a pair $(m, f) : M_A \to N_B$ with $\begin{cases} M \xrightarrow{m} f^\sharp N & \text{in } \mathbf{Mod}_A \\ A \xrightarrow{f} B & \text{in } \mathbf{Alg}_R \end{cases}$, using the restriction of scalars as seen in (2.4). The evident 'forgetful' functor in this case is $V : \mathbf{Mod} \to \mathbf{Alg}_R$, mapping every module M_A to its R-algebra A. Dually to the above results, we get that **Mod** is a complete symmetric monoidal category, with a functor $F : \mathbf{Mod} \to \mathbf{Mod}_R \times \mathbf{Alg}_R$ creating all limits, and also the functor V has the structure of a *strict* monoidal functor, since $V(M_A \otimes N_B) = A \otimes B = VM_A \otimes VN_B$.

Remark 5.1. There is a functor $\operatorname{Hom}(-, N_B) : \mathbf{Comod}^{\mathrm{op}} \to \mathbf{Mod}$ between the categories described above, induced by the bifunctor $\operatorname{Hom} : \mathbf{Comod}_C^{\mathrm{op}} \times \mathbf{Mod}_B \to \mathbf{Mod}_{\operatorname{Hom}_R(C,B)}$ as described in (2.12). More precisely, it is the partial functor of

$$\begin{aligned} \operatorname{Hom} : \mathbf{Comod}^{\mathrm{op}} \times \mathbf{Mod} &\longrightarrow \mathbf{Mod} \\ (X_C, N_B) &\longmapsto \operatorname{Hom}(X, N)_{\operatorname{Hom}_R(C,B)} \end{aligned} \tag{5.2}$$

The diagram

$$\begin{tikzcd} \mathbf{Comod}^{\mathrm{op}} \arrow[r, "\operatorname{Hom}(-, N_B)"] \arrow[d, "G^{\mathrm{op}}"'] & \mathbf{Mod} \arrow[d, "F"] \\ \mathbf{Mod}_R^{\mathrm{op}} \times \mathbf{Coalg}_R^{\mathrm{op}} \arrow[r, "\operatorname{Hom}_R(-,N) \times \operatorname{Hom}_R(-,B)"] & \mathbf{Mod}_R \times \mathbf{Alg}_R \end{tikzcd} \tag{5.3}$$

commutes, the composite on the left side preserves all limits (by (3.2) and Proposition 5.1) and the functor F creates them. Hence $\operatorname{Hom}(-, N_B)$ is a continuous functor.

6. The existence of the universal measuring comodule

Similarly to the way the universal measuring coalgebra was defined via the adjunction (3.4), we now proceed to the definition of an object $Q(M, N)_{P(A,B)}$ in **Comod** defined by a natural isomorphism

$$\mathbf{Comod}(X, Q(M, N)) \cong \mathbf{Mod}(M, \operatorname{Hom}(X, N)), \tag{6.1}$$

where $X = X_C$, $M = M_A$ and $\mathrm{Hom}(X,N) = \mathrm{Hom}(X,N)_{\mathrm{Hom}_R(C,B)}$. Hence, we want to prove the existence of an adjunction

$$\begin{tikzcd} \mathbf{Comod} \arrow[r, shift left, "\mathrm{Hom}(-,N_B)^{\mathrm{op}}"] \arrow[r, phantom, "\perp"] & \mathbf{Mod}^{\mathrm{op}} \arrow[l, shift left, "Q(-,N_B)"] \end{tikzcd} \tag{6.2}$$

where $\mathrm{Hom}(-,N_B)^{\mathrm{op}}$ is the opposite of the functor considered in Remark 5.1.

We begin by considering a slightly different adjunction. First of all, from the special case of (3.4) for $C = P(A,B)$, we get the correspondence

$$\mathbf{Coalg}_R(P(A,B),P(A,B)) \cong \mathbf{Alg}_R(A,\mathrm{Hom}_R(P(A,B),B)),$$

and of course the identity $1_{P(A,B)}$ corresponds uniquely to the unit of the adjunction $\alpha : A \to \mathrm{Hom}_R(P(A,B),B)$. This arrow induces, via the restriction of scalars $\alpha^\sharp : \mathbf{Mod}_{\mathrm{Hom}_R(P(A,B),B)} \to \mathbf{Mod}_A$, a set of morphisms in the category $\mathbf{Mod}_A$, namely $\mathbf{Mod}_A(M,\alpha^\sharp\mathrm{Hom}(Z,N))$, where M is an A-module, Z is a $P(A,B)$-module and N is a B-module. The question then is whether there exists a functor Q and a natural isomorphism

$$\mathbf{Comod}_{P(A,B)}(Z,Q(M,N)) \cong \mathbf{Mod}_A(M,\alpha^\sharp\mathrm{Hom}(Z,N)). \tag{6.3}$$

In other words, we are looking for the left adjoint of the functor

$$\alpha^\sharp \circ \mathrm{Hom}(-,N) : \mathbf{Comod}^{\mathrm{op}}_{P(A,B)} \to \mathbf{Mod}_{\mathrm{Hom}_R(P(A,B),B)} \to \mathbf{Mod}_A.$$

Since the category $\mathbf{Comod}_C$ for any R-coalgebra C is comonadic over $\mathbf{Mod}_R$, cowellpowered and has a generator (see Section 2.4), the Special Adjoint Functor Theorem applies:

- $\mathbf{Comod}^{\mathrm{op}}_{P(A,B)}$ is complete.
- $\mathbf{Comod}^{\mathrm{op}}_{P(A,B)}$ is well-powered.
- $\mathbf{Comod}^{\mathrm{op}}_{P(A,B)}$ has a cogenerator.
- The functor $\alpha^\sharp \circ \mathrm{Hom}(-,N)$ is continuous, as a composite of continuous functors.

Therefore the functor $\alpha^\sharp \circ \mathrm{Hom}(-,N)$ has a left adjoint $Q(-,N)^{\mathrm{op}} : \mathbf{Mod}_A \to \mathbf{Comod}^{\mathrm{op}}_{P(A,B)}$ and (6.3) holds. Moreover, the naturality of the bijection in Z and M makes Q into a bifunctor

$$Q : \mathbf{Mod}^{\mathrm{op}}_A \times \mathbf{Mod}_B \to \mathbf{Comod}_{P(A,B)} \tag{6.4}$$

such that (6.3) is natural in all three variables. We claim that this bifunctor is in fact the one inducing (6.1). More precisely, we are going to show that this functor $Q(-,N)$, when regarded as $Q(-,N_B)$, is also the left adjoint we are after in (6.2).

We will establish a correspondence between elements of $\mathbf{Comod}(X,Q(M,N))$ and elements of $\mathbf{Mod}(M,\mathrm{Hom}(X,N))$, with $M \in \mathbf{Mod}_A$, $N \in \mathbf{Mod}_B$ and $X \in \mathbf{Comod}_C$. As we saw in the previous section, an element of the right hand side $(l,f) : M_A \to \mathrm{Hom}(X,N)_{\mathrm{Hom}_R(C,B)}$ in $\mathbf{Mod}$ is $\begin{cases} M \xrightarrow{l} f^\sharp\mathrm{Hom}(X,N) & \text{in } \mathbf{Mod}_A \\ A \xrightarrow{f} \mathrm{Hom}_R(C,B) & \text{in } \mathbf{Alg}_R \end{cases}$.
This arrow l is a linear map $M \to \mathrm{Hom}_R(X,N)$, which commutes with the coaction of A on both R-modules, the second becoming such via restriction of scalars along f. Explicitly, l satisfies the commutativity of

$$\begin{tikzcd} A \otimes M \arrow[r, "1\otimes l"] \arrow[d, "\mu"'] & A \otimes \mathrm{Hom}(X,N) \arrow[d, dashed] \arrow[r, "f\otimes 1"] & \mathrm{Hom}_R(C,B) \otimes \mathrm{Hom}(X,N) \arrow[dl, "\mu"] \\ M \arrow[r, "l"] & \mathrm{Hom}(X,N) & \end{tikzcd}$$

where $\mathrm{Hom}_R(C,B)\otimes\mathrm{Hom}(X,N)\xrightarrow{\mu}\mathrm{Hom}(X,N)$ is the canonical action on the $\mathrm{Hom}_R(C,B)$-module $\mathrm{Hom}(X,N)$, as described in (2.12). This diagram translates under the adjunction $-\otimes C\dashv\mathrm{Hom}_R(C,-)$ for the adjunct of l, $M\otimes X\xrightarrow{\bar{l}}N$, to

$$\begin{tikzcd}
A\otimes M\otimes X \arrow[r, "1\otimes l\otimes 1"] \arrow[dddd, "\mu\otimes 1"'] & A\otimes\mathrm{Hom}(X,N)\otimes X \arrow[r, "f\otimes 1\otimes 1"] & \mathrm{Hom}_R(C,B)\otimes\mathrm{Hom}(X,N)\otimes X \arrow[d, "1\otimes 1\otimes 1\otimes\delta"] \\
 & & \mathrm{Hom}_R(C,B)\otimes\mathrm{Hom}(X,N)\otimes X\otimes C \arrow[d, "s"] \\
 & & \mathrm{Hom}_R(C,B)\otimes C\otimes\mathrm{Hom}(X,N)\otimes X \arrow[dd, "e\otimes e"] \\
 & & \\
M\otimes X \arrow[dr, "\bar{l}"] & & B\otimes N \arrow[dl, "\mu"] \\
 & N &
\end{tikzcd}$$

using (2.9). This can also be written as

$$\begin{tikzcd}
A\otimes M\otimes X \arrow[r, "1\otimes l\otimes\delta"] \arrow[ddd, "\mu\otimes 1"'] & A\otimes\mathrm{Hom}(X,N)\otimes X\otimes C \arrow[r, "f\otimes 1"] \arrow[ddr, dashed] \arrow[dr, phantom, "(*)"] & \mathrm{Hom}_R(C,B)\otimes\mathrm{Hom}(X,N)\otimes X\otimes C \arrow[d, "s"] \\
 & & \mathrm{Hom}_R(C,B)\otimes C\otimes\mathrm{Hom}(X,N)\otimes X \arrow[d, "e\otimes 1\otimes 1"] \\
 & & B\otimes\mathrm{Hom}(X,N)\otimes X \arrow[d, "1\otimes e"] \\
M\otimes X \arrow[dr, "\bar{l}"'] & & B\otimes N \arrow[dl, "\mu"] \\
 & N &
\end{tikzcd} \tag{6.5}$$

The question is whether every element of the left hand side of (6.1), when the functor Q is the same as in (6.4), corresponds uniquely to a linear map with the properties of l as described above. Note that, by definition of Q, when $M_A,N_B\in\mathbf{Comod}$, $Q(M,N)$ is a $P(A,B)$-comodule. So, a morphism $X_C\xrightarrow{(k,h)}Q(M,N)_{P(A,B)}$ in $\mathbf{Comod}$ is a pair $\begin{cases} h_*X\xrightarrow{k}Q(M,N) & \text{in }\mathbf{Comod}_{P(A,B)} \\ C\xrightarrow{h}P(A,B) & \text{in }\mathbf{Coalg}_R \end{cases}$.

By the adjunction (3.4) defining the universal measuring coalgebra, we already know that each one of these coalgebra maps h can be written as $\hat{f}$ for a unique $f:A\to\mathrm{Hom}_R(C,B)$ in $\mathbf{Alg}_R$.

The key point is that, since $h_*X\equiv\hat{f}_*X$ becomes a $P(A,B)$-comodule via corestriction of scalars along $\hat{f}$, the map $k:\hat{f}_*X\to Q(M,N)$ in $\mathbf{Comod}_{P(A,B)}$ is an element of the left hand side of the special case adjunction (6.3), for $Z=\hat{f}_*X$. Therefore it uniquely corresponds to some $t:M\to\alpha^\sharp\mathrm{Hom}(\hat{f}_*X,N)$ in $\mathbf{Mod}_A$. We will show that this t, which as a linear map is $M\xrightarrow{t}\mathrm{Hom}_R(X,N)$, has the property described by the commutativity of (6.5) above, and hence it is an element of the right hand side of (6.1).

We first have to see in detail how $\alpha^\sharp\mathrm{Hom}(\hat{f}_*X,N)$ has the structure of an A-module, with underlying R-module $\mathrm{Hom}_R(X,N)$. The A-action is given by

$$\begin{tikzcd}
A\otimes\mathrm{Hom}(X,N) \arrow[r, "\alpha\otimes 1"] \arrow[ddr, dashed, "\mu'"'] & \mathrm{Hom}_R(P(A,B),B)\otimes\mathrm{Hom}(X,N) \arrow[d, "\mathrm{Hom}(\hat{f},1)\otimes 1"] \\
 & \mathrm{Hom}_R(C,B)\otimes\mathrm{Hom}(X,N) \arrow[d, "\mu"] \\
 & \mathrm{Hom}(X,N)
\end{tikzcd}$$

which corresponds under the adjunction $-\otimes C \dashv \mathrm{Hom}_R(C,-)$ to

$$\begin{tikzcd}
A\otimes \mathrm{Hom}(X,N)\otimes X \arrow[r, "\alpha\otimes 1\otimes 1"] \arrow[ddddr, squiggly, "\bar{\mu}'"'] & \mathrm{Hom}_R(P(A,B),B)\otimes \mathrm{Hom}(X,N)\otimes X \arrow[d, "1\otimes 1\otimes \delta"] \\
& \mathrm{Hom}_R(P(A,B),B)\otimes \mathrm{Hom}(X,N)\otimes X\otimes C \arrow[d, "1\otimes 1\otimes 1\otimes \hat{f}"] \\
& \mathrm{Hom}_R(P(A,B),B)\otimes \mathrm{Hom}(X,N)\otimes X\otimes P(A,B) \arrow[d, "1\otimes s"] \\
& \mathrm{Hom}_R(P(A,B),B)\otimes P(A,B)\otimes \mathrm{Hom}(X,N)\otimes X \arrow[d, "e\otimes e"] \\
& N.
\end{tikzcd}$$

So, the regular diagram which the linear map $t: M\to \mathrm{Hom}_R(X,N)$ as an A-module map has to satisfy (see (2.3))

$$\begin{tikzcd}
A\otimes M \arrow[r, "1\otimes t"] \arrow[d, "\mu"'] & A\otimes \mathrm{Hom}(X,N) \arrow[d, "\mu'"] \\
M \arrow[r, "t"] & \mathrm{Hom}(X,N)
\end{tikzcd}$$

corresponds under $-\otimes C \dashv \mathrm{Hom}_R(C,-)$, for its adjunct $\bar{t}: M\otimes X\to N$, to

$$\begin{tikzcd}
A\otimes \mathrm{Hom}(X,N)\otimes X\otimes C \arrow[rr, "\alpha\otimes 1"] \arrow[dddrr, dashed] & & \mathrm{Hom}(P(A,B),B)\otimes \mathrm{Hom}(X,N)\otimes X\otimes C \arrow[d, "1\otimes 1\otimes 1\otimes \hat{f}"] \\
& (**) & \mathrm{Hom}(P(A,B),B)\otimes \mathrm{Hom}(X,N)\otimes X\otimes P(A,B) \arrow[d, "1\otimes s"] \\
A\otimes M\otimes X \arrow[uu, bend right, "1\otimes t\otimes \delta"'] \arrow[dd, bend left, "\mu\otimes 1"] & & \mathrm{Hom}(P(A,B),B)\otimes P(A,B)\otimes \mathrm{Hom}(X,N)\otimes X \arrow[d, "e\otimes 1\otimes 1"] \\
& & B\otimes \mathrm{Hom}(X,N)\otimes X \arrow[d, "1\otimes e"] \\
M\otimes X \arrow[dr, "\bar{t}"'] & & B\otimes N \arrow[dl, "\mu"'] \\
& N &
\end{tikzcd} \tag{6.6}$$

Hence, in order for the linear map $t: M\to \mathrm{Hom}_R(X,N)$ to be an element of the right hand side of (6.1), like the map l described earlier, we just have to show that the diagrams (6.5) and (6.6) are actually the same. By inspection of the two diagrams, we see that it suffices to show that the parts (*) and (**) are the same. In other words, since the factor $\mathrm{Hom}(X,N)\otimes X$ remains unchanged, the diagram

$$\begin{tikzcd}
& \mathrm{Hom}_R(P(A,B),B)\otimes C \arrow[r, "1\otimes \hat{f}"] & \mathrm{Hom}_R(P(A,B),B)\otimes P(A,B) \arrow[dr, "e"] & \\
A\otimes C \arrow[ur, "\alpha\otimes 1"] \arrow[drr, "f\otimes 1"'] & & & B \\
& & \mathrm{Hom}_R(C,B)\otimes C \arrow[ur, "e"'] &
\end{tikzcd}$$

must commute. But this is satisfied by Lemma 3.1, so the proof is complete.

To summarise, for the arbitrary element of $\mathbf{Comod}(X_C, Q(M,N)_{P(A,B)})$ that we started with, $\begin{cases} h_*X \xrightarrow{k} Q(M,N) & \text{in } \mathbf{Comod}_{P(A,B)} \\ C \xrightarrow{h} P(A,B) & \text{in } \mathbf{Coalg}_R \end{cases}$, we already knew that h

corresponds uniquely to some algebra map $f : A \to \mathrm{Hom}_R(C, B)$, and moreover we showed that k corresponds uniquely to some $t : M \to \alpha^\sharp \mathrm{Hom}(\hat{f}_* X, N)$ in $\mathbf{Mod}_A$, which is a linear map $M \to \mathrm{Hom}_R(X, N)$ with exactly the same properties as an A-module map $M \to f^\sharp \mathrm{Hom}(X, N)$. Therefore we established a (natural) correspondance

$$\mathbf{Comod}(X, Q(M, N)) \cong \mathbf{Mod}(M, \mathrm{Hom}(X, N))$$

which gives the adjunction (6.2). Notice that the bifunctor Q, defined initially as in (6.4), is used in this context as

$$\begin{aligned} Q : \mathbf{Mod}^{\mathrm{op}} \times \mathbf{Mod} &\longrightarrow \mathbf{Comod} \\ (M_A, N_B) &\longmapsto Q(M, N)_{P(A,B)} \end{aligned} \tag{6.7}$$

We call the object $Q(M, N)$ the *universal measuring comodule*. As mentioned earlier, $Q(M_A, N_B)$ is a $P(A, B)$-comodule by construction.

Remark 6.1. Using the established adjunctions $\mathrm{Hom}(-, N_B)^{\mathrm{op}} \dashv Q(-, N_B)$ and $\mathrm{Hom}_R(-, B)^{\mathrm{op}} \dashv P(-, B)$, we have a diagram of categories and functors

$$\begin{tikzcd} \mathbf{Comod} \arrow[r, shift left, "\mathrm{Hom}(-,N_B)^{\mathrm{op}}"] \arrow[r, phantom, "\perp"] \arrow[d, "U"'] & \mathbf{Mod}^{\mathrm{op}} \arrow[l, shift left, "Q(-,N_B)"] \arrow[d, "V^{\mathrm{op}}"] \\ \mathbf{Coalg}_R \arrow[r, shift left, "\mathrm{Hom}_R(-,B)^{\mathrm{op}}"] \arrow[r, phantom, "\perp"] & \mathbf{Alg}_R^{\mathrm{op}} \arrow[l, shift left, "P(-,B)"] \end{tikzcd}$$

where the squares of right adjoints/left adjoints respectively serially commute.

7. Enrichment of modules in comodules

In a similar way to Section 4, we will see how the bifunctor $Q : \mathbf{Mod}^{\mathrm{op}} \times \mathbf{Mod} \to \mathbf{Comod}$ induces an enrichment of the category $\mathbf{Mod}$ in the symmetric monoidal category $\mathbf{Comod}$. We will again use Proposition 2.2 in the forms of Corollaries 2.2 and 2.3, for $\mathcal{C} = \mathbf{Comod}$ and $\mathcal{A} = \mathbf{Mod}$.

Consider the bifunctor $\mathrm{Hom} : \mathbf{Comod}^{\mathrm{op}} \times \mathbf{Mod} \to \mathbf{Mod}$ as in (5.2). In the previous section we showed the existence of an adjunction $\begin{tikzcd} \mathbf{Comod} \arrow[r, shift left, "\mathrm{Hom}(-,N_B)^{\mathrm{op}}"] \arrow[r, phantom, "\perp"] & \mathbf{Mod}^{\mathrm{op}} \arrow[l, shift left, "Q(-,N_B)"] \end{tikzcd}$.
So again, for the conditions of Proposition 2.2 to be satisfied, we only need to show that the functor $\mathrm{Hom} = H$ in this case is an action of the monoidal $\mathbf{Comod}^{\mathrm{op}}$ on $\mathbf{Mod}$. We have to find two natural isomorphisms $\alpha_{XYN} : \mathrm{Hom}(X \otimes Y, N) \xrightarrow{\sim} \mathrm{Hom}(X, \mathrm{Hom}(Y, N))$ and $\lambda_N : \mathrm{Hom}(R, N) \xrightarrow{\sim} N$ in $\mathbf{Mod}$ such that diagrams (2.15), (2.16) and (2.17) commute. We already know that these isomorphisms exist in $\mathbf{Mod}_R$ as linear maps, so we have to make sure that they can be lifted to isomorphisms in $\mathbf{Mod}$, as explained below.

The first isomorphism in $\mathbf{Mod}_R$, for $X \in \mathbf{Comod}_C$, $Y \in \mathbf{Comod}_D$ and $N \in \mathbf{Mod}_A$, is as a linear map

$$\begin{aligned} k : \mathrm{Hom}_R(X, \mathrm{Hom}_R(Y, N)) &\longrightarrow \mathrm{Hom}_R(X \otimes Y, N) \\ f : X \to \mathrm{Hom}_R(Y, N) &\longmapsto [k(f)] : X \otimes Y \to N \\ &\qquad [k(f)](x \otimes y) := [f(x)](y) \end{aligned}$$

The domain is a $\mathrm{Hom}_R(C, \mathrm{Hom}_R(D, A))$-module and the codomain is a $\mathrm{Hom}_R(C \otimes D, A)$-module in a canonical way. As mentioned in Section 4, there is an isomorphism $\beta : \mathrm{Hom}_R(C, \mathrm{Hom}_R(D, A)) \to \mathrm{Hom}_R(C \otimes D, A)$ in $\mathbf{Alg}_R$. Then, it can be

seen that k commutes with the $\mathrm{Hom}_R(C, \mathrm{Hom}_R(D, A))$-actions on both modules, the first one on $\mathrm{Hom}_R(X, \mathrm{Hom}_R(Y, N))$ being the canonical (see (2.9)) and the second on $\mathrm{Hom}_R(X \otimes Y, N)$ induced via restriction of scalars along β. Therefore, we obtain an isomorphism in **Mod**, given by

$$\begin{cases} \mathrm{Hom}(X, \mathrm{Hom}(Y, N)) \xrightarrow{m} \beta^\sharp \mathrm{Hom}(X \otimes Y, N) & \text{in } \mathbf{Mod}_{\mathrm{Hom}_R(C, \mathrm{Hom}_R(D,A))} \\ \mathrm{Hom}_R(C, \mathrm{Hom}_R(D, A)) \xrightarrow{\beta} \mathrm{Hom}_R(C \otimes D, A) & \text{in } \mathbf{Alg}_R \end{cases}.$$

With similar calculations, we obtain the second isomorphism λ in **Mod**, and the diagrams (2.15), (2.16) and (2.17) commute because they do for all R-modules. Hence, the bifunctor Hom is an action, and so its opposite $\mathrm{Hom}^{\mathrm{op}} : \mathbf{Comod} \times \mathbf{Mod}^{\mathrm{op}} \to \mathbf{Mod}^{\mathrm{op}}$ is also an action of the symmetric monoidal category **Comod** on the category $\mathbf{Mod}^{\mathrm{op}}$.

Now the assumptions of Corollaries 2.2 and 2.3 hold, so we obtain the following results.

Proposition 7.1. *The category* $\mathbf{Mod}^{\mathrm{op}}$ *is enriched in* **Comod**, *with hom-objects* $\mathbf{Mod}^{\mathrm{op}}(M, N) = Q(N, M)$.

Proposition 7.2. *The category* **Mod** *is enriched in* **Comod**, *with hom-objects* $\mathbf{Mod}(M, N) = Q(M, N)$.

8. The category of comodules revisited

When we proved the existence of the bifunctor $Q : \mathbf{Mod}^{\mathrm{op}} \times \mathbf{Mod} \to \mathbf{Comod}$ as the (parametrized) adjoint of $\mathrm{Hom}^{\mathrm{op}} : \mathbf{Comod} \times \mathbf{Mod}^{\mathrm{op}} \to \mathbf{Mod}^{\mathrm{op}}$, we first considered a 'special case' of this adjunction (see (6.3)) between the specific fibers $\mathbf{Comod}_{P(A,B)}$ and $\mathbf{Mod}_A$, and then we used it to prove the more general adjunction between **Comod** and **Mod**. Using a very similar idea, we can show that the symmetric monoidal category **Comod** is monoidal closed.

Recall that, by Proposition 2.1, $\mathbf{Coalg}_R$ is a monoidal closed category, via the adjunction

$$\begin{tikzcd} \mathbf{Coalg}_R \arrow[r, shift left, "-\otimes D"] \arrow[r, phantom, "\perp"] & \mathbf{Coalg}_R. \arrow[l, shift left, "{[D,-]_c}"] \end{tikzcd} \tag{8.1}$$

The identity morphism $1_{[D,E]}$ corresponds uniquely to the counit of this adjunction $\varepsilon_E : [D, E]_c \otimes D \to E$, and this arrow induces the corestriction of scalars $\varepsilon_* : \mathbf{Comod}_{[D,E]_c \otimes E} \to \mathbf{Comod}_E$ between the respective fibers. So we can again consider the existence of a 'special case' adjunction: for $Y \in \mathbf{Comod}_D$, the functor

$$\varepsilon_* \circ (- \otimes Y) : \mathbf{Comod}_{[D,E]_c} \to \mathbf{Comod}_{[D,E]_c \otimes E} \to \mathbf{Comod}_E$$

has a right adjoint $H(Y, -) : \mathbf{Comod}_E \to \mathbf{Comod}_{[D,E]_c}$, since again $\mathbf{Comod}_{[D,E]_c}$ is cocomplete, co-wellpowered, has a generator and both ε_* and $(- \otimes Y)$ preserve colimits. We obtain a natural isomorphism

$$\mathbf{Comod}_E(\varepsilon_*(W \otimes Y), Z) \cong \mathbf{Comod}_{[D,E]_c}(W, H(Y, Z)) \tag{8.2}$$

where $W \in \mathbf{Comod}_{[D,E]_c}$, $Y \in \mathbf{Comod}_D$ and $Z \in \mathbf{Comod}_E$.

In order to prove that **Comod** is monoidal closed, we are after a more general adjunction $\begin{tikzcd} \mathbf{Comod} \arrow[r, shift left, "-\otimes Y_D"] \arrow[r, phantom, "\perp"] & \mathbf{Comod} \arrow[l, shift left, "{H(Y_D,-)}"] \end{tikzcd}$ with a natural isomorphism

$$\mathbf{Comod}(X \otimes Y, Z) \cong \mathbf{Comod}(X, H(Y, Z)). \tag{8.3}$$

for $X \in \mathbf{Comod}_C$. As in Section 6, we claim that the right adjoint H of the adjunction (8.2), when seen as $H : \mathbf{Comod}^{\mathrm{op}} \times \mathbf{Comod} \to \mathbf{Comod}$, is the same

one inducing this bijection. So, in order to show the correspondance between arrows

$$\begin{cases} f_*(X\otimes Y)\xrightarrow{k} Z & \text{in } \mathbf{Comod}_E \\ C\otimes D\xrightarrow{f} E & \text{in } \mathbf{Coalg}_R \end{cases} \quad \text{and} \quad \begin{cases} \bar{f}_*X\xrightarrow{l} H(Y,Z) & \text{in } \mathbf{Comod}_{[D,E]_c} \\ C\xrightarrow{\bar{f}} [D,E]_c & \text{in } \mathbf{Coalg}_R \end{cases},$$

where $\bar{f}$ is the transpose of f under (8.1), we can start from the right hand side and take the corresponding

$$\Big\{\bar{f}_*X\xrightarrow{l} H(Y,Z) \text{ in } \mathbf{Comod}_{[D,E]_c}\Big\} \overset{(8.2)}{\Longleftrightarrow} \Big\{\varepsilon_*(\bar{f}_*X\otimes Y)\xrightarrow{m} Z \text{ in } \mathbf{Comod}_E\Big\},$$

and the commutative diagram which expresses the fact that the linear map $m: X\otimes Y\to Z$ commutes with the induced E-action on $\varepsilon_*(\bar{f}_*X\otimes Y)$ turns out to be the same with the diagram expressing the fact that m is a E-comodule map $f_*(X\otimes Y)\xrightarrow{m} Z$, using that f and $\bar{f}$ are transpose maps. Therefore (8.3) holds, and the bifunctor

$$\begin{aligned} H: \mathbf{Comod}^{\mathrm{op}}\times\mathbf{Comod} &\longrightarrow \mathbf{Comod} \\ (Y_D, Z_E) &\longmapsto H(Y,Z)_{[D,E]_c} \end{aligned} \tag{8.4}$$

is the internal hom of **Comod**. Note that $H(Y_D,Z_E)$ is a $[D,E]_c$-comodule by construction.

Proposition 8.1. *The global category of comodules* **Comod** *is a monoidal closed category.*

The functor $-\otimes Y_D : \mathbf{Comod}\to\mathbf{Comod}$ whose right adjoint is the internal hom of **Comod**, is evidently cocontinuous, by the commutativity of

$$\begin{tikzcd} \mathbf{Comod} \arrow[r, "-\otimes Y_D"] \arrow[d, "G"'] & \mathbf{Comod} \arrow[d, "G"] \\ \mathbf{Mod}_R\times\mathbf{Coalg}_R \arrow[r, "(-\otimes Y)\times(-\otimes D)"] & \mathbf{Mod}_R\otimes\mathbf{Coalg}_R \end{tikzcd} \tag{8.5}$$

where G is comonadic and the bottom arrow preserves colimits since $\mathbf{Mod}_R$ and $\mathbf{Coalg}_R$ are monoidal closed.

Remark 8.1. When $R=k$ is a field, the existence of the universal measuring comodule and the internal hom in **Comod** become clearer. More precisely, rather than showing how the correspondences (6.1), (8.3) are established explicitly, we can prove the existence of right adjoints of the functors $\mathrm{Hom}(-,N_B)^{\mathrm{op}}:\mathbf{Comod}\to\mathbf{Mod}^{\mathrm{op}}$ and $-\otimes Y_D:\mathbf{Comod}\to\mathbf{Comod}$ directly, using Theorem 2.1 by Kelly.

We already know that **Comod** is a cocomplete category (Corollary 5.1) and the functors $\mathrm{Hom}(-,N_B)^{\mathrm{op}}$ and $-\otimes Y_D$ are cocontinuous (Remark 5.1 and (8.5)). If we can show that **Comod** has a small dense subcategory, both functors will automatically have right adjoints, establishing the existence of the functors Q and H.

Consider the category of all finite dimensional (as vector spaces) comodules over finite dimensional coalgebras and linear maps between them, denoted $\mathcal{C}_{f.d.}$. The proof that $\mathcal{C}_{f.d.}$ is a (small) dense subcategory of **Comod** makes use of the notion of the *coefficients coalgebra* $\mathrm{Coeff}(X)$ over a finite dimensional C-comodule X (see [1] as the space of matrix coefficients), which is the smallest subcoalgebra of C such that X is a $\mathrm{Coeff}(X)$-comodule, and even more it is a finite dimensional coalgebra.

Sketch of proof. For a fixed $X_C\in\mathbf{Comod}$, consider the category

$$\Lambda=\{(V,D)\,\epsilon\, Sub_f(X)\times Sub_f(C)/\ \mathrm{Coeff}(V)\subset D\}$$

where $Sub_f(C)$ is the preorder of finite dimensional subcoalgebras of C and $Sub_f(X)$ the preorder of finite dimensional subcomodules of the comodule X. For each $(V,D) \in \Lambda$, since V is a subcomodule of X_C, $V \in \mathbf{Comod}_C$, but we can restrict the coaction to $D \subset C$ and we denote the D-comodule V_D. Then, we can define a diagram

$$\begin{aligned} \phi : \quad \Lambda &\longrightarrow \mathbf{Comod} \\ (V,D) &\longmapsto V_D \end{aligned}$$

and we have a cocone $(V_D \xrightarrow{(\tau_{(V,D)},g)} X_C \,/\, (V,D) \in \Lambda)$ in $\mathbf{Comod}$, where g is the inclusion $g = \iota_D : D \hookrightarrow C$ in $\mathbf{Coalg}_k$ and $\tau_{(V,D)} : (\iota_D)_* V_D \to X$ is again the inclusion $j_V : V \hookrightarrow X$ in $\mathbf{Comod}_C$.

The Fundamental Theorem of Coalgebras and the Fundamental Theorem of Comodules state that C is the colimit of its f.d. subcoalgebras and X is the colimit of its f.d. subcomodules, and then we can show that both g and $\tau_{(V,D)}$ are colimiting cocones in $\mathbf{Coalg}_k$ and $\mathbf{Comod}_C$ respectively. Therefore $(V_D \xrightarrow{(\tau_{(V,D)},g)} X_C \,/\, (V,D) \in \Lambda)$ is a colimiting cocone in $\mathbf{Comod}$.

In order to prove that $\mathcal{C}_{f.d}$ is dense in $\mathbf{Comod}$, it suffices to show that the diagram

$$(I \downarrow X_C) \xrightarrow{P} \mathcal{C}_{f.d} \xhookrightarrow{I} \mathbf{Comod}$$

has X_C as its canonical colimit, for any $X_C \in \mathbf{Comod}$. For that, we form the diagram

$$\begin{tikzcd} \Lambda \arrow[r, "F"] \arrow[ddr, "\phi"'] & (I \downarrow X_C) \arrow[d, "P"] \\ & \mathcal{C}_{f.d.} \arrow[d, "I"] \\ & \mathbf{Comod} \end{tikzcd}$$

which commutes, and we can prove that the functor F is final, using epi-mono factorizations and finiteness properties of coalgebras and comodules. We already know the colimiting cocone for the diagram ϕ, therefore every comodule is the canonical colimit of elements of $\mathcal{C}_{f.d.}$ and the proof is complete. $\square$

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